The 2PI-loop expansion revisited

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in collaboration with

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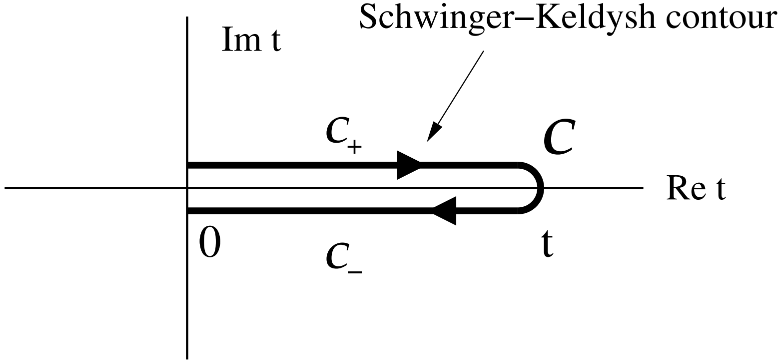
Quantum fields out of equilibrium

Out of equilibrium, the density matrix and operator expectation values evolve in time, ($\mathcal{Z} = Tr[\rho] = 1$)

$$\langle \mathcal{O}(t) \rangle_\rho = Tr[\rho(t)\mathcal{O}] = Tr[\rho(0)\, e^{-iHt}\, \mathcal{O}\, e^{iHt}].$$

We can write this as a path-integral, as an evolution along a path in complex t-space,



$$\mathcal{Z}[\rho, \mathcal{C}, J_i, K] = \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \langle \phi_+ | \rho(0) | \phi_- \rangle$$

$$\times \int \mathcal{D}\phi \exp \left\{ i \left(S_{\mathcal{C}}[\phi] + J\,\phi + \frac{1}{2}\phi\, K\,\phi \right) \right\},$$

$$S_{\mathcal{C}}[\phi] + J\phi + \frac{1}{2}\phi K \phi = \int_{\mathcal{C}} dt \int d^3x [\mathcal{L}(\phi) + J_{\pm}\phi + \frac{1}{2}\phi K \phi].$$

The currents $J_{\pm}$ and K live on upper (+)/lower (-) branches, and $\phi(\mathbf{x}, 0_+) = \phi_+, \phi(\mathbf{x}, 0_-) = \phi_-$. The initial condition in $\langle \phi_+ | \rho(0) | \phi_- \rangle$ can be included in the currents.

Approximate quantum dynamics

Φ-derivable approximations defined in terms of truncated loop expansions of the 2PI effective action provide a framework for approximating the quantum dynamics (in and) out of equilibrium.

Equations of motion are derived in terms of the 1- and 2-point functions (the mean field $\bar{\phi}$ and the propagator G). The derived equations of motion have to be solved numerically, by discretizing the field on a lattice.

[Berges&Cox:2000, Aarts&Berges:2001, Berges:2002, Aarts et al.:2002, Cassing et al.:2003, Berges et al.:2004], thermalization for scalar fields in 1+1, 2+1 and 3+1 dimensions, [Berges et al:2003] fermions and scalars in 3+1 dimensions. [Berges&Serreau:2003, Arrizabalaga et al.(AT):2004] for studies of preheating. For gauge fields, see [Arrizabalaga&Smit:2003, Berges:2004]. [van Hees&Knoll:2002, Blaizot et al.:2003, Berges et al.:2005], renormalization].

2PI effective action

$$Z[J,K] = \int d\phi\, e^{i\,S[\phi]+i\,J\,\phi+\frac{i}{2}\phi\,K\,\phi} = e^{i\,W[J,K]},$$

$$\bar{\phi}(x) = \frac{\delta W[J,K]}{\delta J(x)}, \quad G(x,y) = 2\frac{\delta W[J,K]}{\delta K(x,y)} - \bar{\phi}(x)\bar{\phi}(y).$$

Define:

$$\Gamma^{2PI}[\bar{\phi},G] = W[J] - J\,\bar{\phi} - \frac{1}{2}K\,\left(G+\bar{\phi}\bar{\phi}\right).$$

Then:

$$\frac{\delta\Gamma^{2PI}[\bar{\phi},G]}{\delta\bar{\phi}(x)} = -J(x) - \int d^4y\,K(x,y)\bar{\phi}(y),$$

$$\frac{\delta\Gamma^{2PI}[\bar{\phi},G]}{\delta G(x,y)} = -\frac{1}{2}K(x,y).$$

Φ-derivable approximations

The 2PI effective action can be written as [Cornwall et al :1974]:

$$\Gamma^{2PI}[\bar{\phi}, G] = S[\bar{\phi}] - \frac{i}{2}\mathrm{Tr}\,\ln G + \frac{i}{2}\mathrm{Tr}\,G_0^{-1}G + \Phi[\bar{\phi}, G],$$

with

$$iG_0^{-1}(x,y) = \frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}(x)\delta\bar{\phi}(y)}\delta(x-y).$$

Φ is the sum of all 2PI skeleton diagrams with no external lines, full propagators and bare vertices. A choice of truncation of this series $\Phi_{\mathrm{tr}} \to \Gamma^{2PI}_{\mathrm{tr}}$ constitutes a Φ-derived approximation. Equations of motion derived from any truncation

$$\frac{\delta\Gamma^{2PI}_{\mathrm{tr}}[\bar{\phi}, G]}{\delta G(x,y)} = 0, \qquad \frac{\delta\Gamma^{2PI}_{\mathrm{tr}}[\bar{\phi}, G]}{\delta\bar{\phi}(x)} = 0$$

conserve an energy functional.

Equations of motion

The equations of motion for a homogeneous system :

$$
\begin{aligned}
\partial_t^2 F(t,t',\mathbf{x}) = \partial_i^2 F(t,t',\mathbf{x}) &- M_{\rm eff}^2(t) F(t,t',\mathbf{x}) \\
&+ \int_0^t dt''\, d\mathbf{y}\, \Sigma_\rho(t,t'',\mathbf{y}) F(t'',t',\mathbf{x}-\mathbf{y}) \\
&- \int_0^{t'} dt''\, d\mathbf{y}\, \Sigma_F(t,t'',\mathbf{y}) \rho(t'',t',\mathbf{x}-\mathbf{y}), \\
\partial_t^2 \rho(t,t',\mathbf{x}) = \partial_i^2 \rho(t,t',\mathbf{x}) &- M_{\rm eff}^2(t) \rho(t,t',\mathbf{x}) \\
&+ \int_{t'}^t dt''\, d\mathbf{y}\, \Sigma_F(t,t'',\mathbf{y}) \rho(t'',t',\mathbf{x}-\mathbf{y}), \\
\partial_t^2 \bar\phi(t) = -M_{\rm eff}^2(t)\bar\phi(t) &- \int_0^t dt''\, d\mathbf{y}\, \Sigma_\phi(t,t'',\mathbf{y}) \bar\phi(t'').
\end{aligned}
$$

with $F = \langle \{\phi,\phi\}\rangle$, $\rho = i\langle[\phi,\phi]\rangle$. These are the Kadanoff-Baym equations (if non-relativistic) [Kadanoff&Baym:1962].

ϕ^4-model

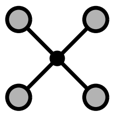
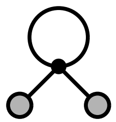
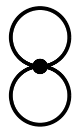
We study a single real scalar field in a ϕ^4 potential:

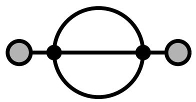
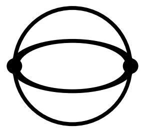
$$S = -\int dt \int d^3x \left[\frac{\partial_\mu \phi \, \partial^\mu \phi}{2} + \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 \right] .$$

In the symmetric phase $m^2 > 0$, $\langle \phi \rangle \simeq 0$ and in the broken phase $m^2 < 0$, $\langle \phi \rangle \simeq \sqrt{6|m^2|/\lambda}$.

The functional Φ

The functional Φ in a loop expansion is

$$\Phi[\bar{\phi}, G] = \frac{1}{4!}$$  $$+ \frac{1}{2}$$  $$+ \frac{1}{8}$$ 

$$+ \frac{1}{12}$$  $$+ \frac{1}{48}$$  $$+ \ldots$$

Motivation

- Thermalization for scalar fields in 3+1d.
- Most 2PI studies compare $O(\lambda)$ (Hartree) to $O(\lambda^2)$ (Basketball) or LO-1/N to NLO-1/N. $\langle\phi\rangle = 0$.
- Hartree/LO-1/N: Leading contribution to $\mathrm{Re}\Sigma$. Basketball/NLO-1/N: Next-to-Leading contribution to $\mathrm{Re}\Sigma$.
- Basketball/NLO-1/N: Leading contribution to $\mathrm{Im}\Sigma$. No results exist at NNLO. Numerically Intractable(?).
- The $\mathrm{Im}\Sigma$ is related to decay/damping rates, equilibration times.
- $\langle\phi\rangle \neq 0$ induces three point vertex, and an additional diagram at $O(\lambda)$. Perturbatively, $\mathrm{Im}\Sigma = 0$ (on shell). 2PI-resummed?
- Three point vertex present in non-abelian gauge theories.

Initial conditions

We apply two initialization schemes, Top-hat and Thermal, determined by a choice for the distribution function n_k and the dispersion relation ω_k, through:

$$\langle \phi_k \phi_{-k} \rangle = \frac{n_k + 1/2}{\omega_k}, \quad \langle \pi_k \pi_{-k} \rangle = (n_k + 1/2)\omega_k,$$

- Top-hat:

$$n_k = 0; \qquad n_k = c > 0, \quad k_{\min} < k < k_{\max},$$

for some $c > 0$.

- Thermal:

$$n_k = (e^{\omega_k/T} - 1)^{-1}.$$

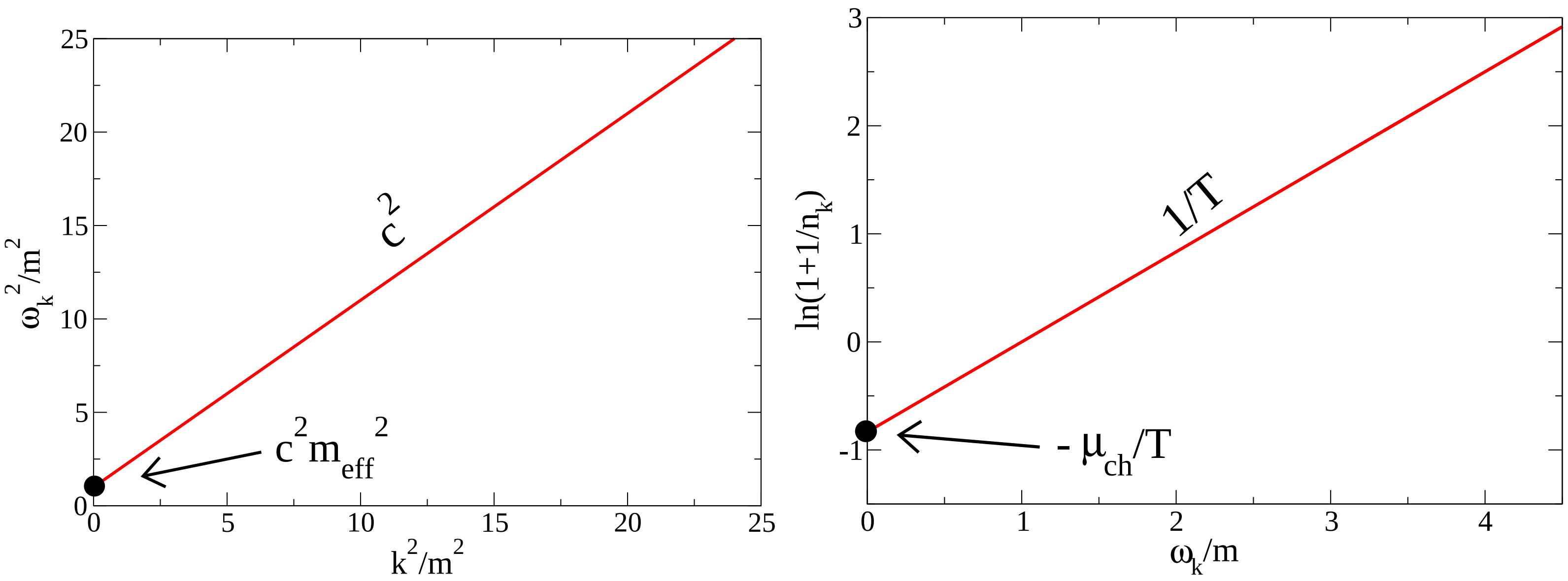
In both cases, $\omega_k^2 = k^2 + m^2$.

What equilibrium looks like

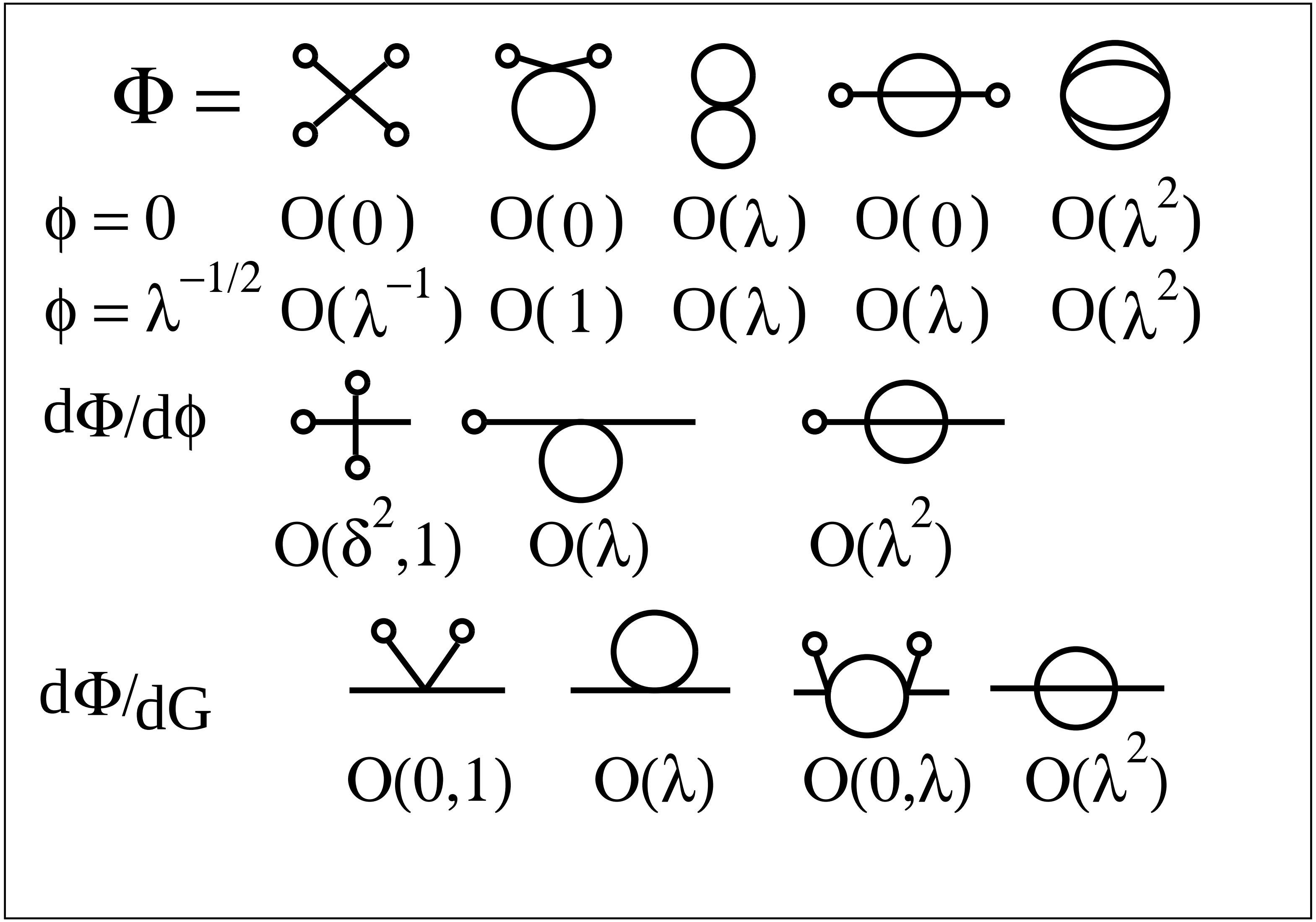
For small coupling we expect the equilibrium state to have a quasi-particle dispersion relation and a Bose-Einstein particle distribution. We will define (also out of equilibrium)

$$\begin{aligned} \omega_k \quad &= \sqrt{\langle \pi_k \pi_{-k} \rangle / \langle \phi_k \phi_{-k} \rangle}, \\ n_k + 1/2 \quad &= \sqrt{\langle \pi_k \pi_{-k} \rangle \langle \phi_k \phi_{-k} \rangle}. \end{aligned}$$

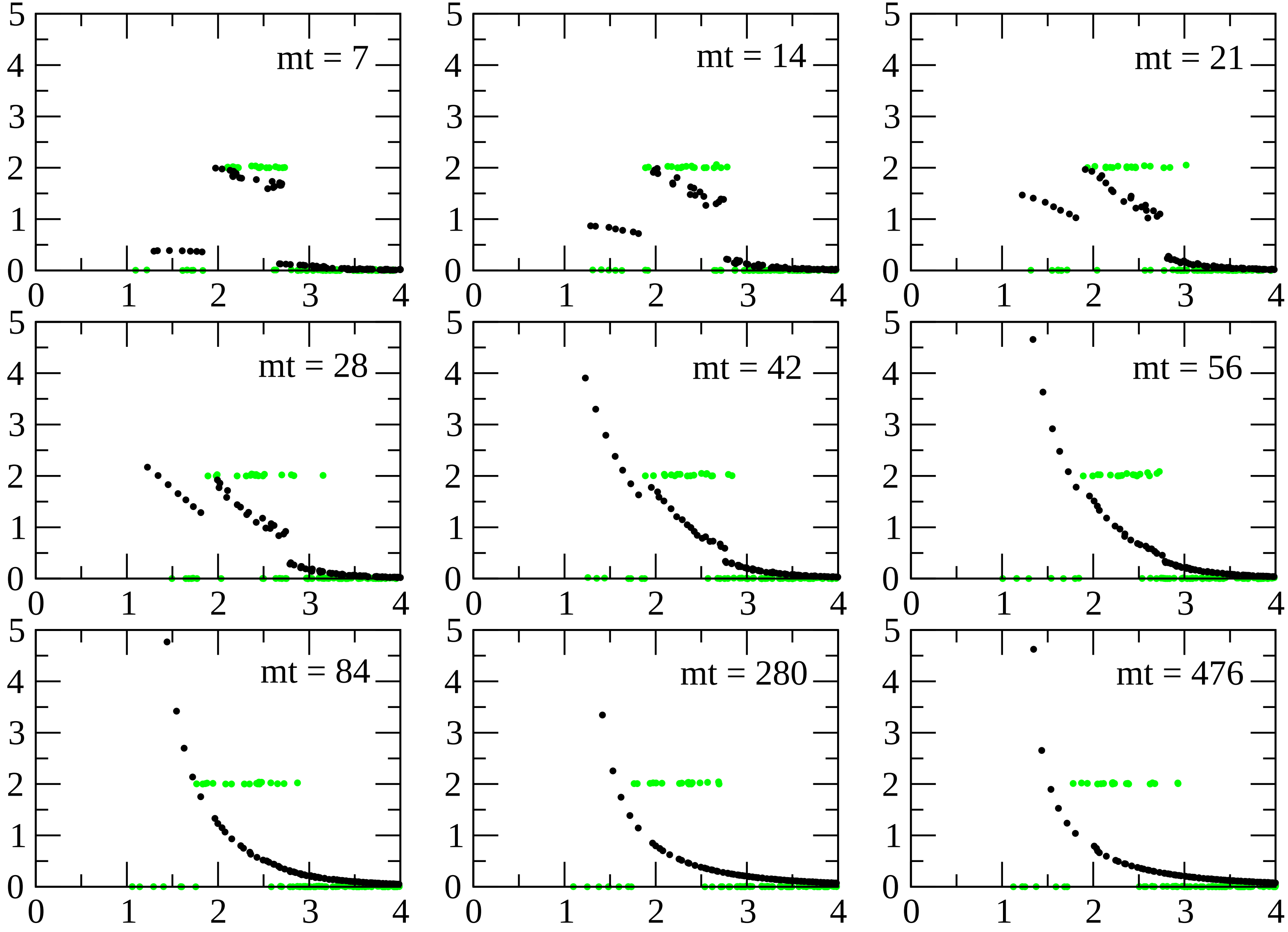
When plotting ω_k^2 vs. k^2 the slope is c^2 and the intercept $c^2 m_{\mathrm{eff}}^2(T)$. In a plot of $\ln(1 + 1/n_k)$ vs. ω_k, the slope is T^{-1} and the intercept $-\mu_{\mathrm{ch}}/T$.



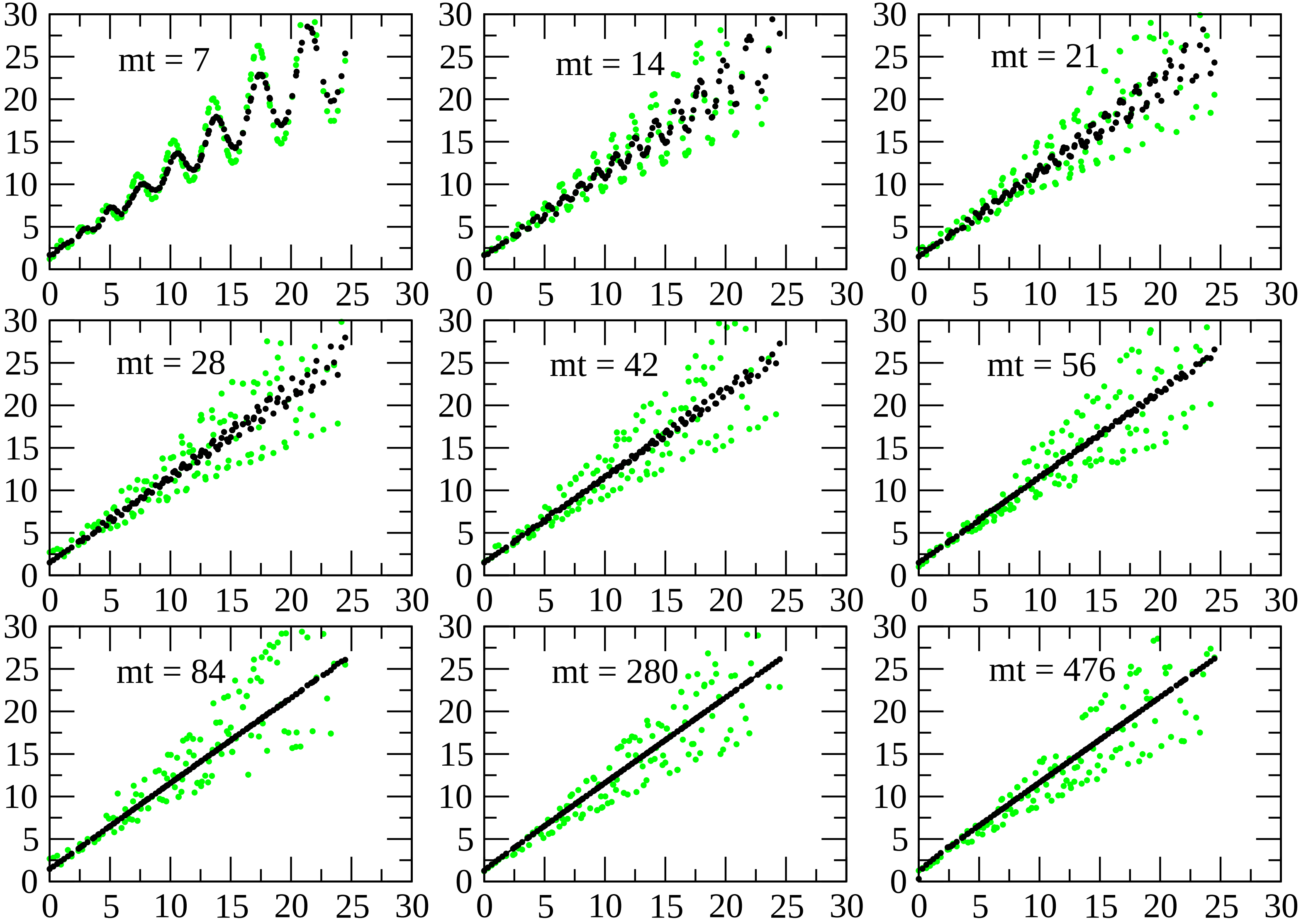
25
20
15
10
5
0
0
5
10
15
20
25
ω_k^2/m^2
k^2/m^2
c^2
$c^2 m_{eff}^2$
3
2
1
0
-1
0
1
2
3
4
$\ln(1+1/n_k)$
ω_k/m
1/T
$-\mu_{ch}/T$


$\Phi =$
$\phi = 0$ $O(0)$ $O(0)$ $O(\lambda)$ $O(0)$ $O(\lambda^2)$
$\phi = \lambda^{-1/2}$ $O(\lambda^{-1})$ $O(1)$ $O(\lambda)$ $O(\lambda)$ $O(\lambda^2)$
$d\Phi/d\phi$
$O(\delta^2,1)$ $O(\lambda)$ $O(\lambda^2)$
$d\Phi/dG$
$O(0,1)$ $O(\lambda)$ $O(0,\lambda)$ $O(\lambda^2)$

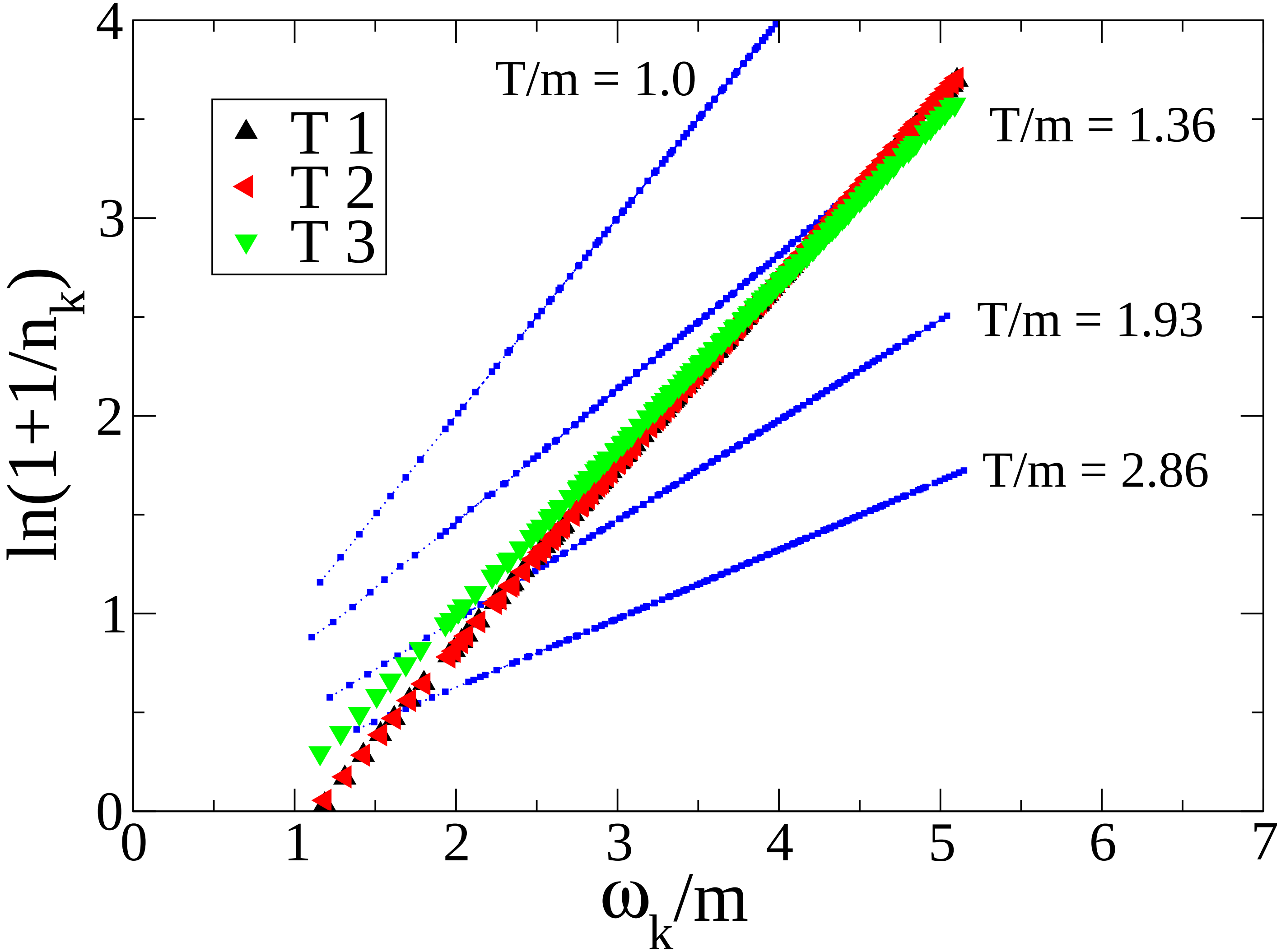
Symmetric phase, including basketball

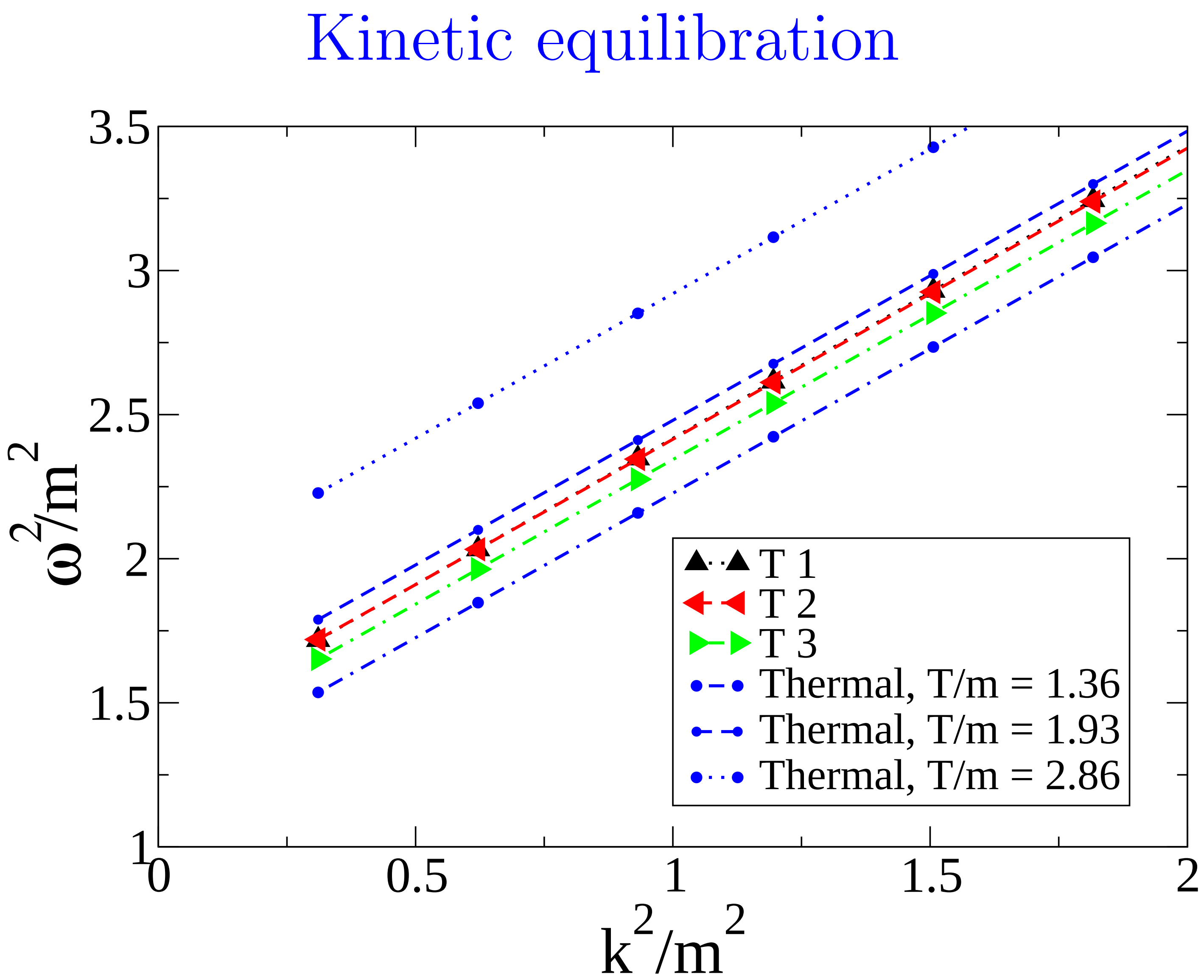


Symmetric phase, including basketball

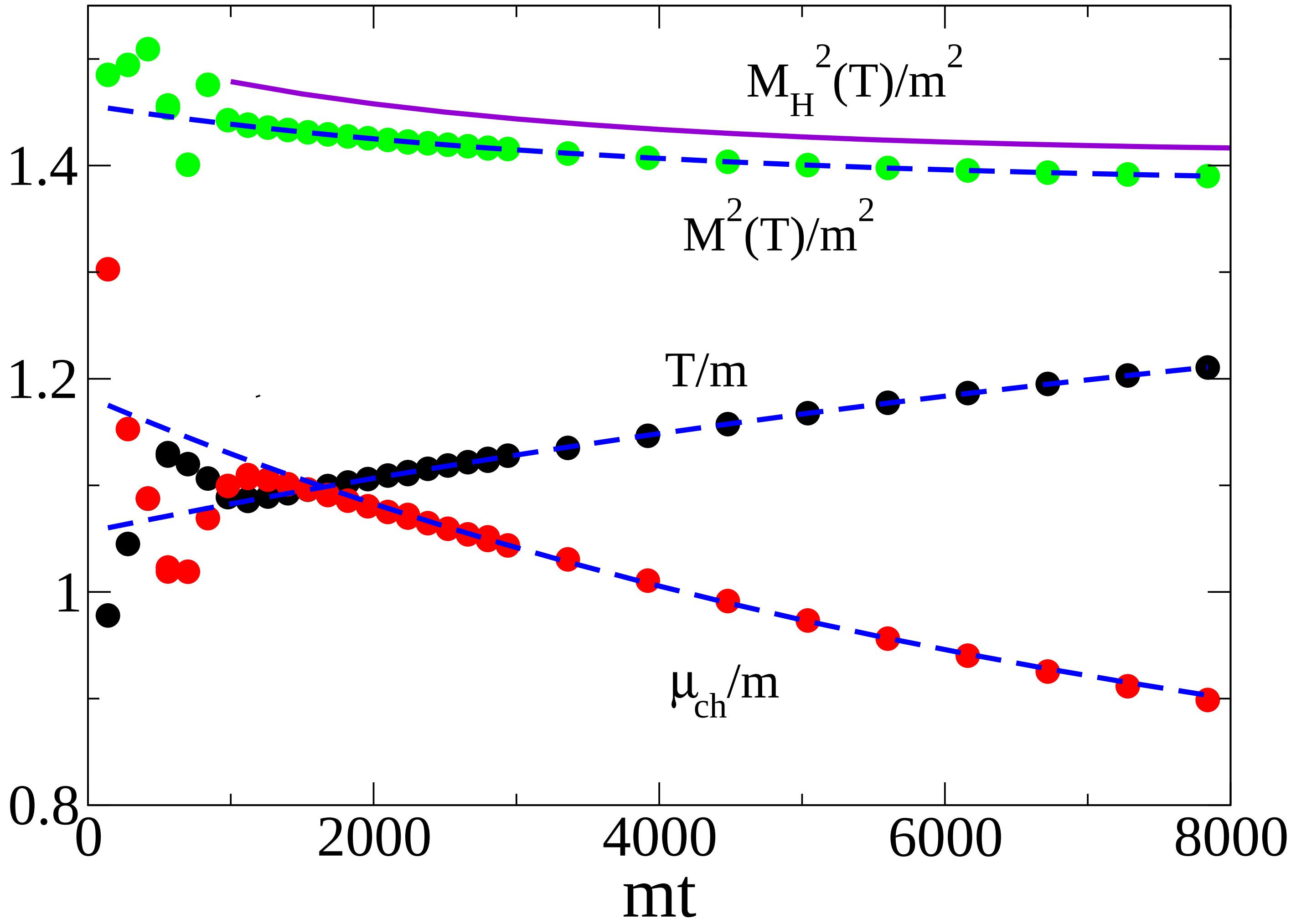


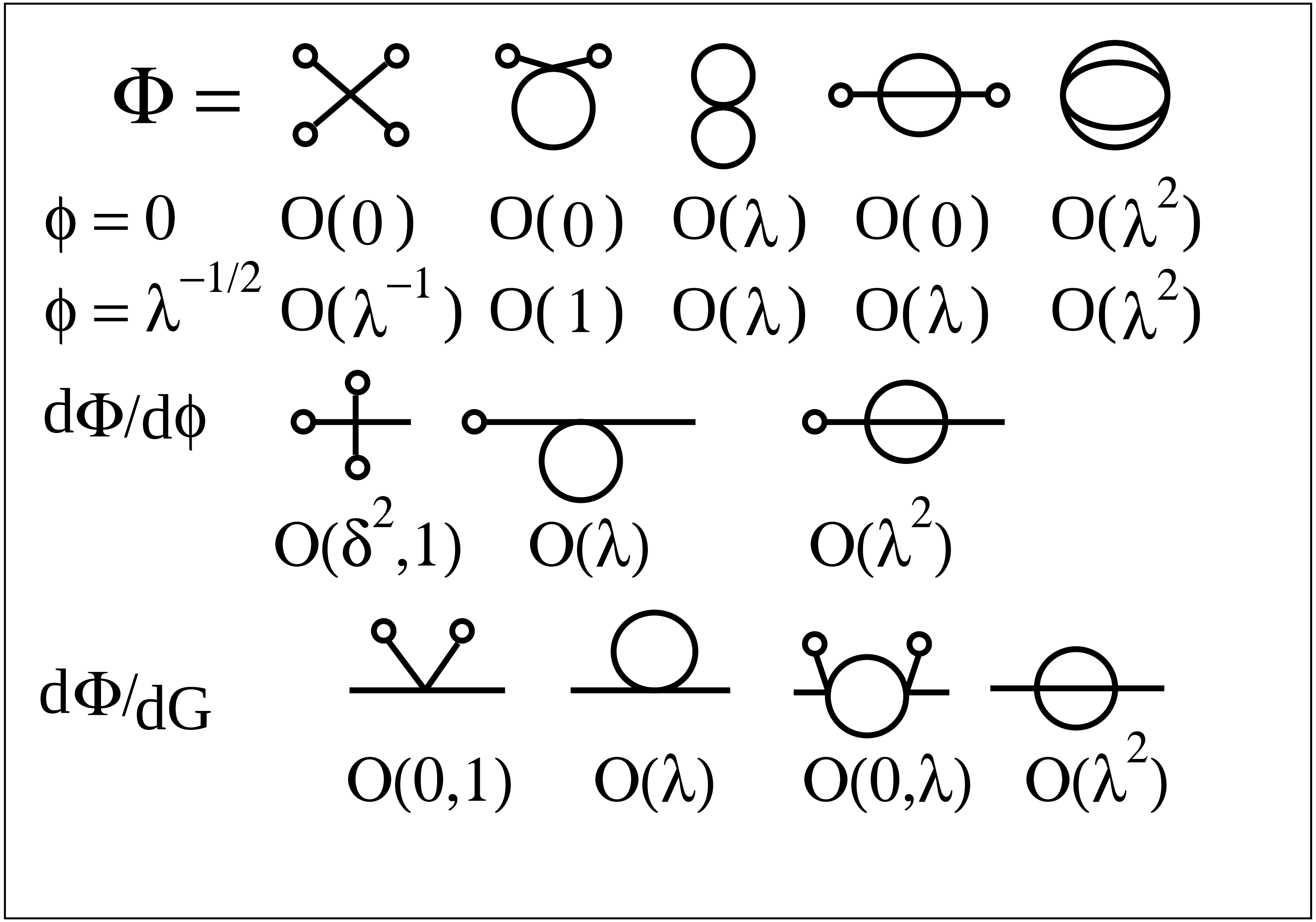
Kinetic equilibration




Kinetic equilibration
3.5
3
2.5
2
1.5
1
0
0.5
1
1.5
2
ω²/m²
k²/m²
T 1
T 2
T 3
Thermal, T/m = 1.36
Thermal, T/m = 1.93
Thermal, T/m = 2.86

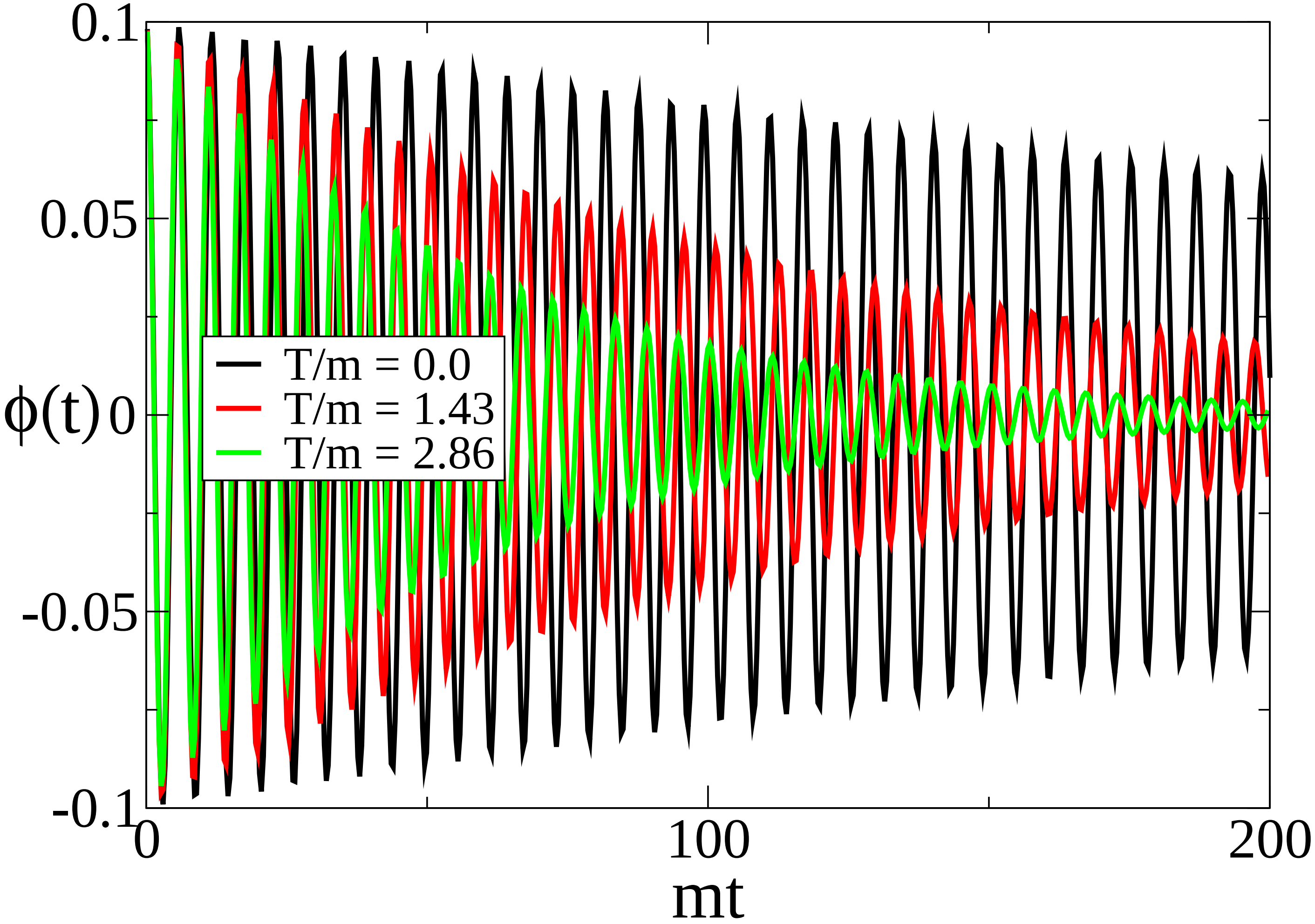
Chemical equilibration

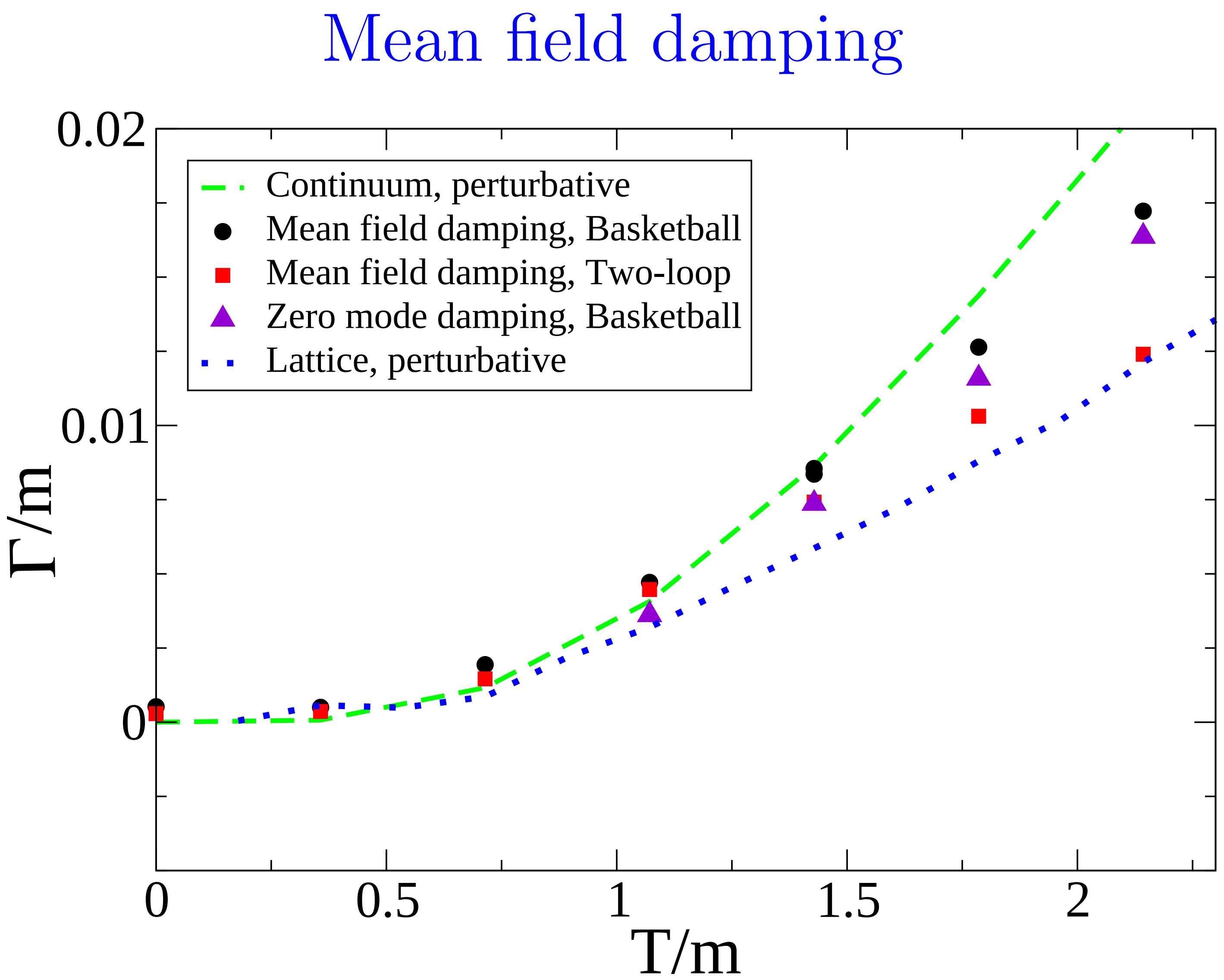


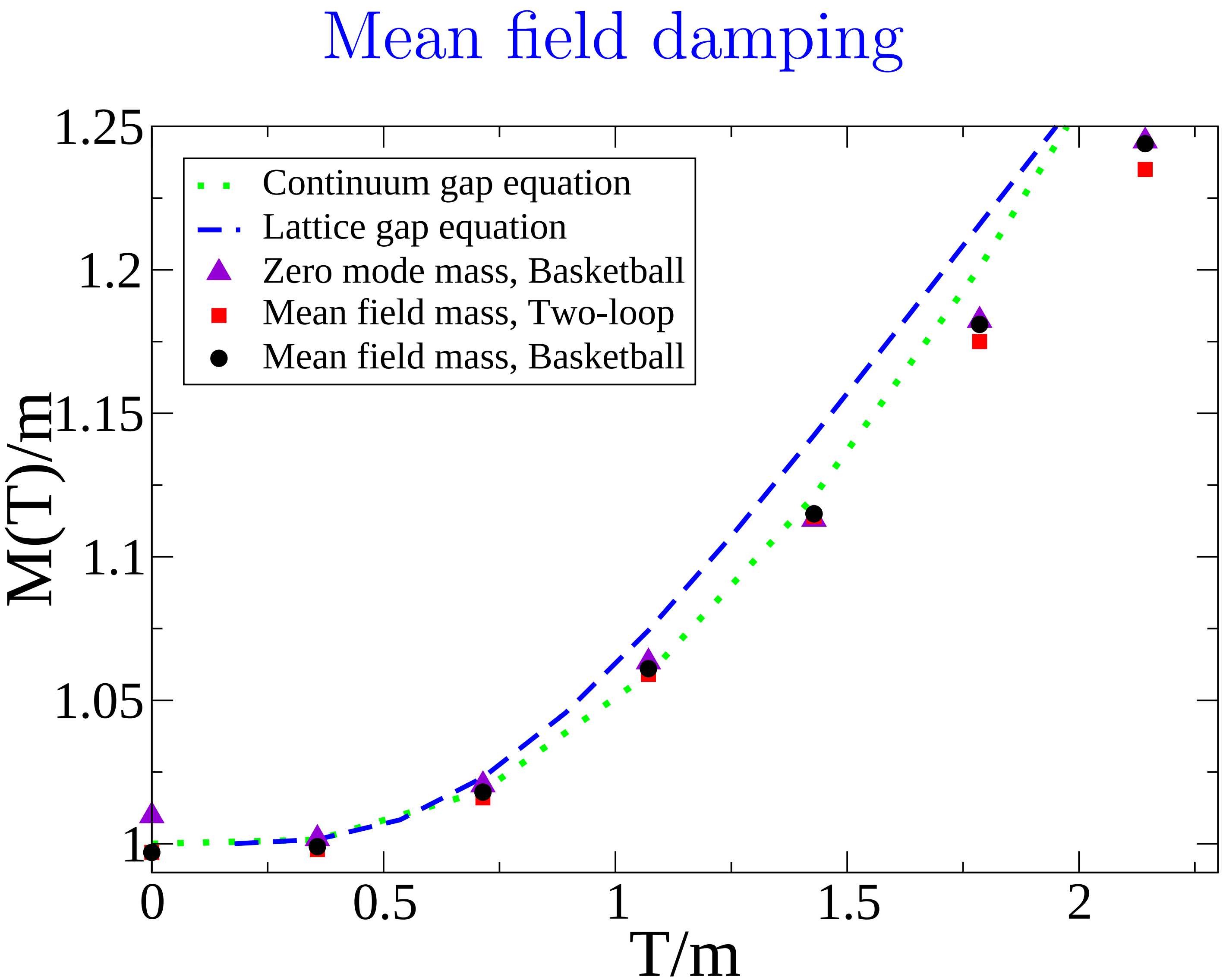


Φ =
φ = 0
O(0) O(0) O(λ) O(0) O(λ²)
φ = λ^−1/2
O(λ^−1) O(1) O(λ) O(λ) O(λ²)
dΦ/dφ
O(δ²,1) O(λ) O(λ²)
dΦ/dG
O(0,1) O(λ) O(0,λ) O(λ²)

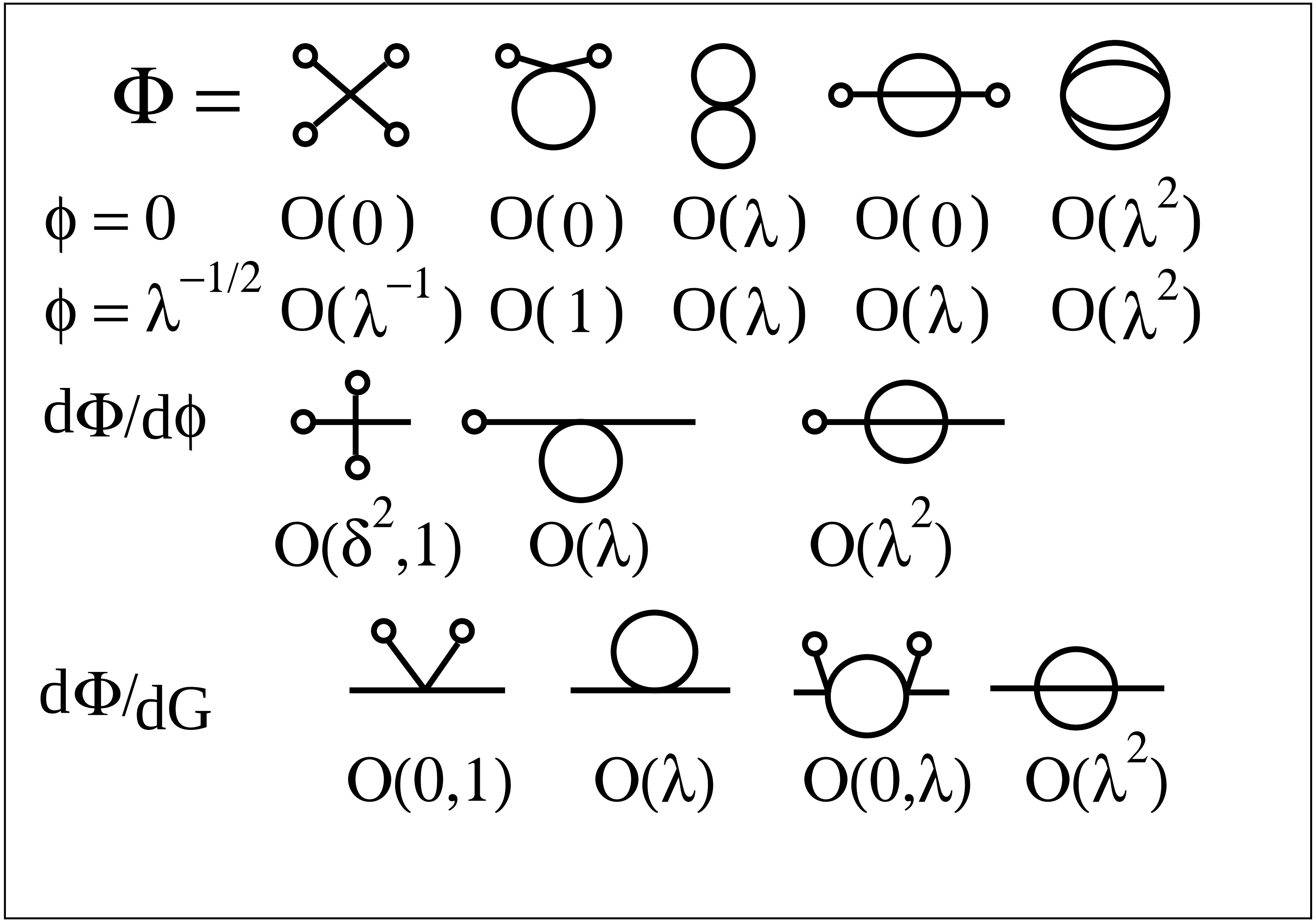
Mean field damping



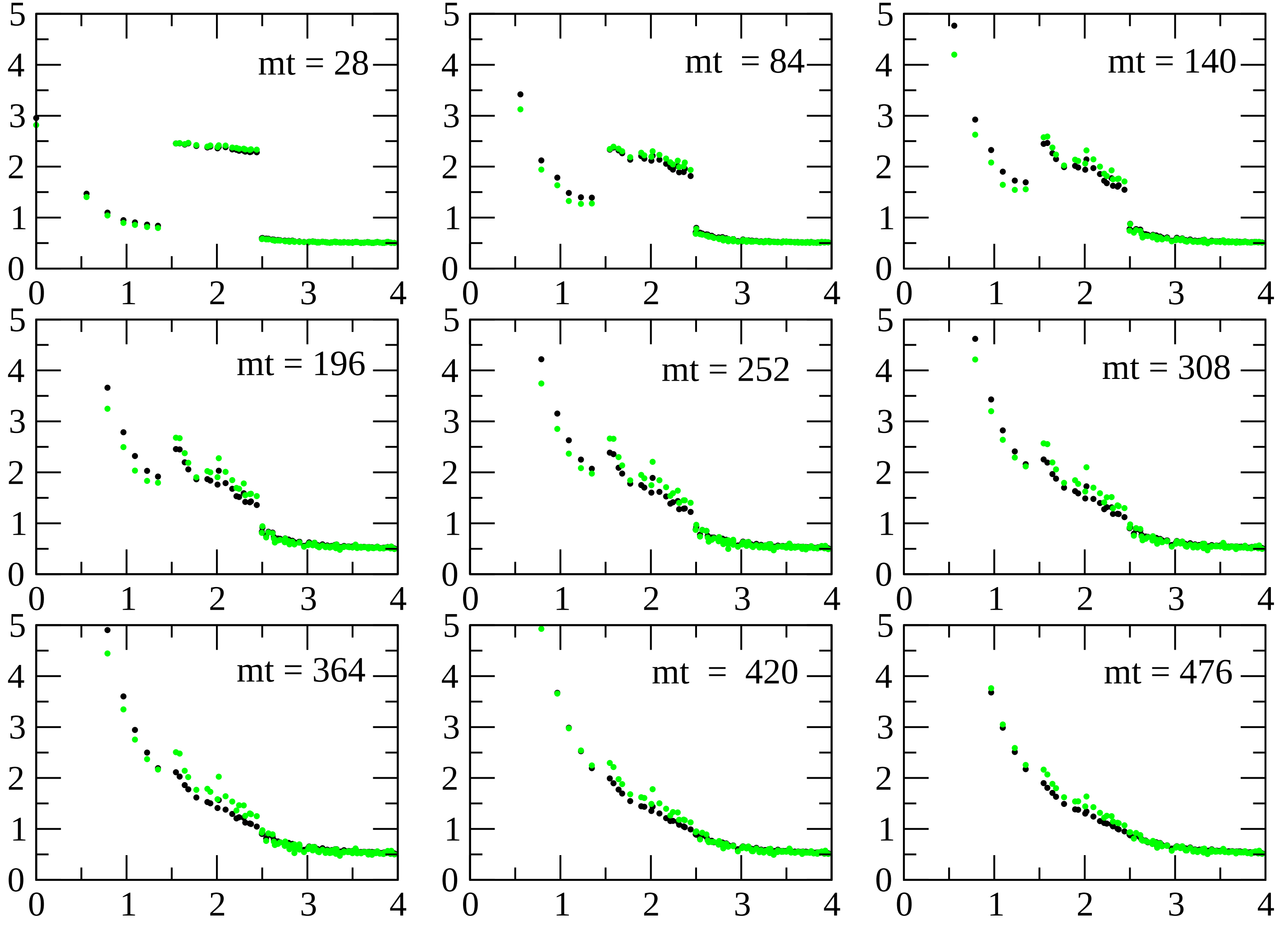

Mean field damping
Continuum, perturbative
Mean field damping, Basketball
Mean field damping, Two-loop
Zero mode damping, Basketball
Lattice, perturbative
0.02
0.01
0
Γ/m
0
0.5
1
1.5
2
T/m



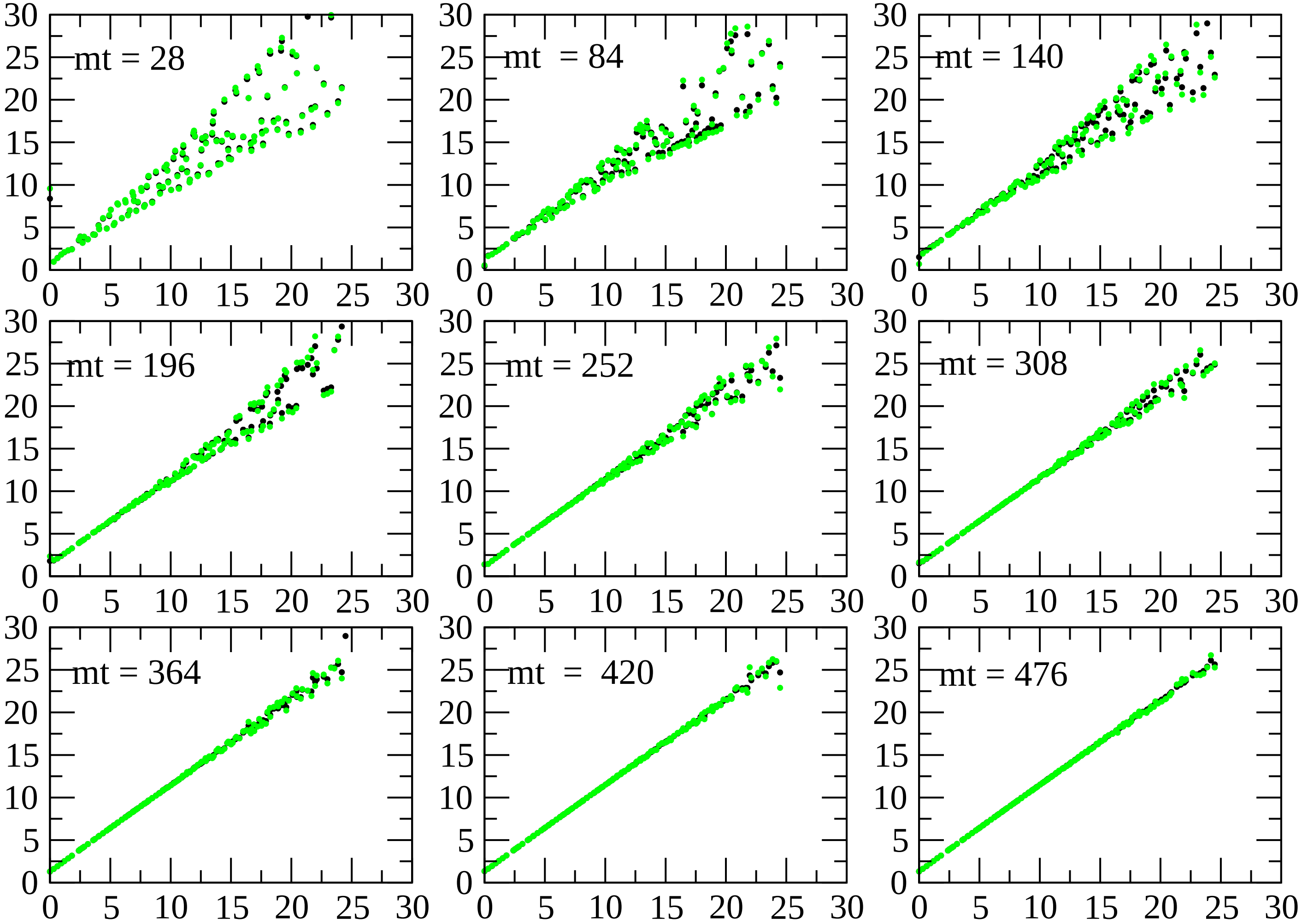
Mean field damping
Continuum gap equation
Lattice gap equation
Zero mode mass, Basketball
Mean field mass, Two-loop
Mean field mass, Basketball
M(T)/m
T/m
1.25
1.2
1.15
1.1
1.05
1
0
0.5
1
1.5
2


Φ =
φ = 0
O(0) O(0) O(λ) O(0) O(λ²)
φ = λ^(−1/2)
O(λ^(−1)) O(1) O(λ) O(λ) O(λ²)
dΦ/dφ
O(δ²,1) O(λ) O(λ²)
dΦ/dG
O(0,1) O(λ) O(0,λ) O(λ²)

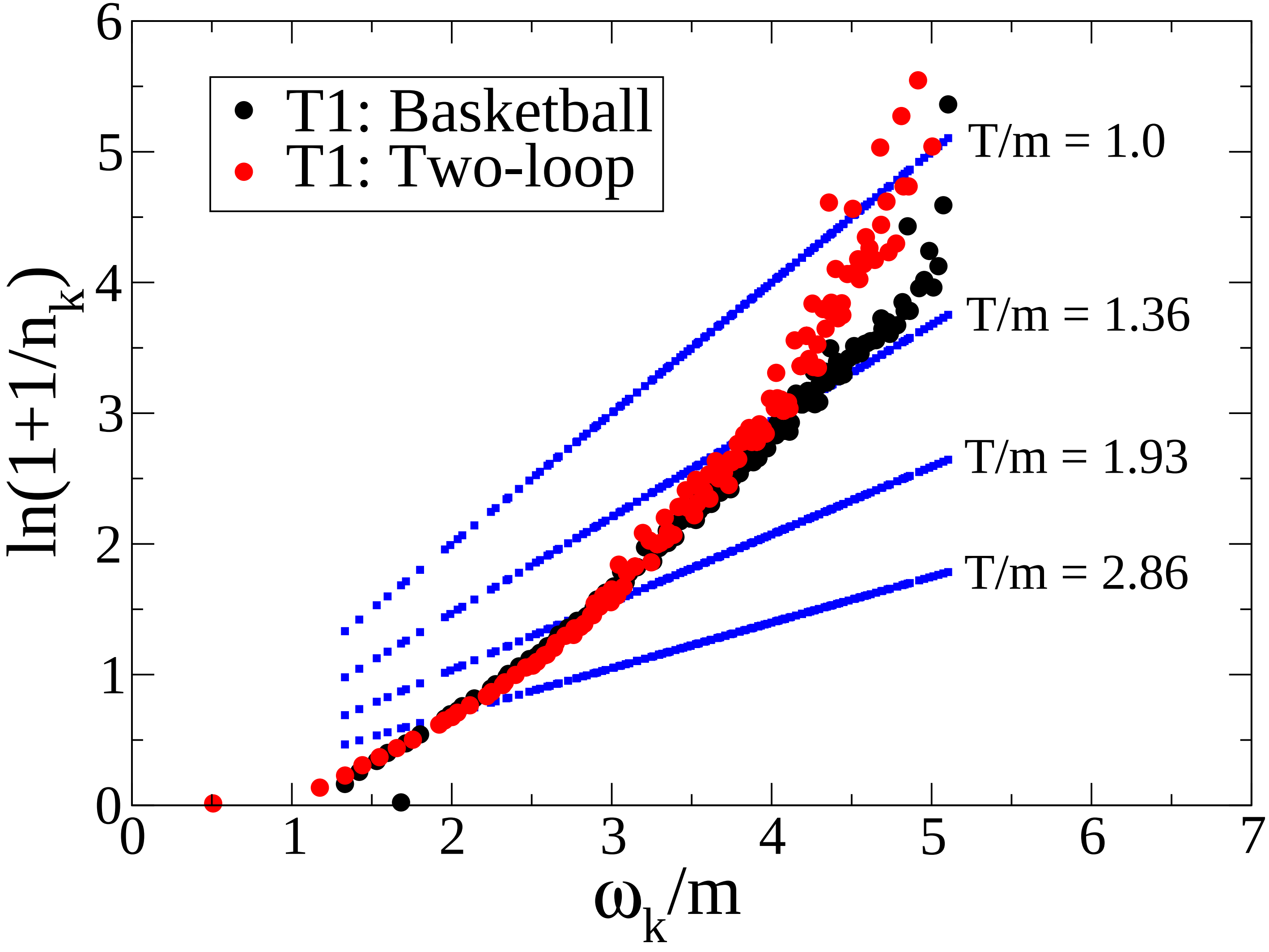
Broken phase



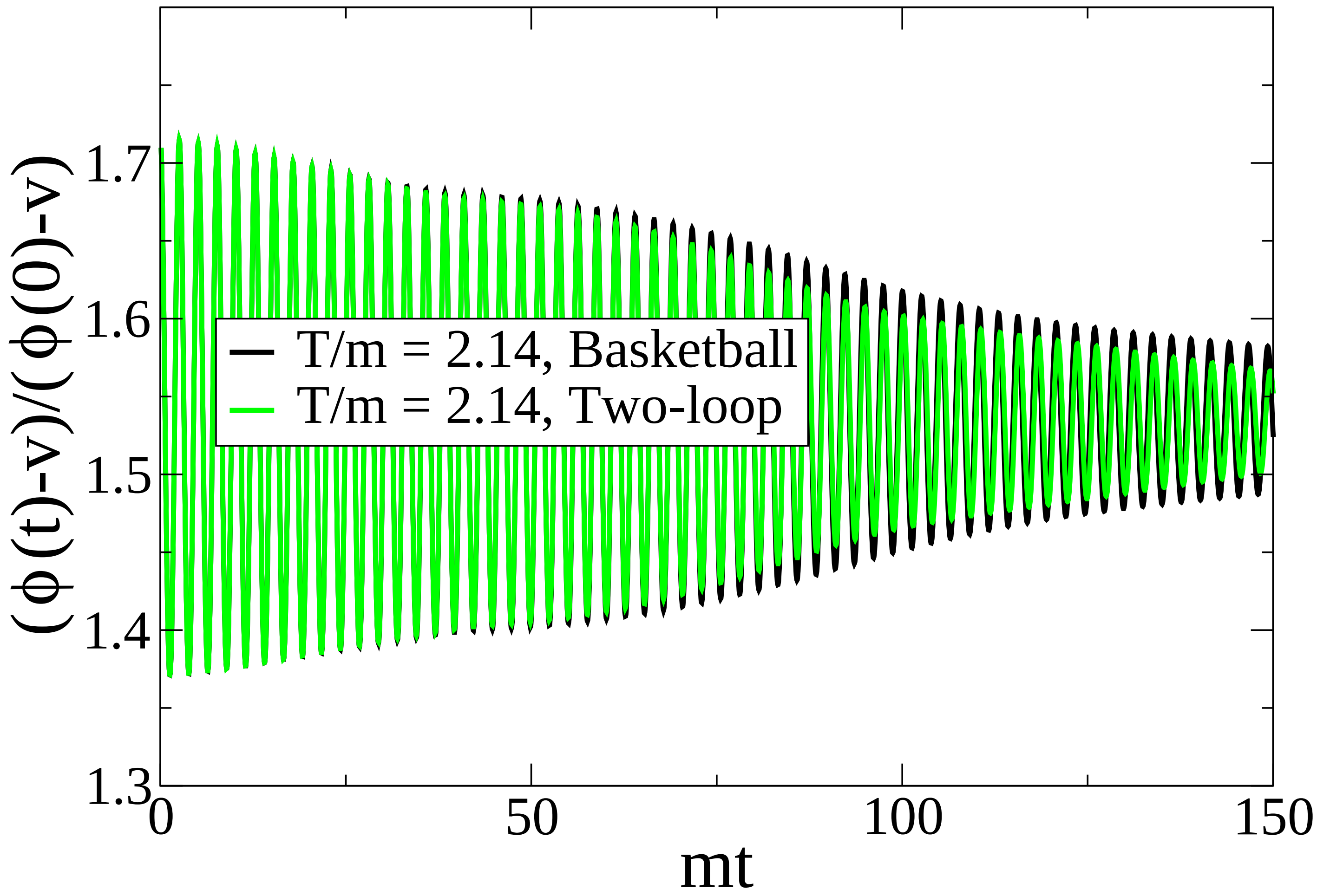
Broken phase

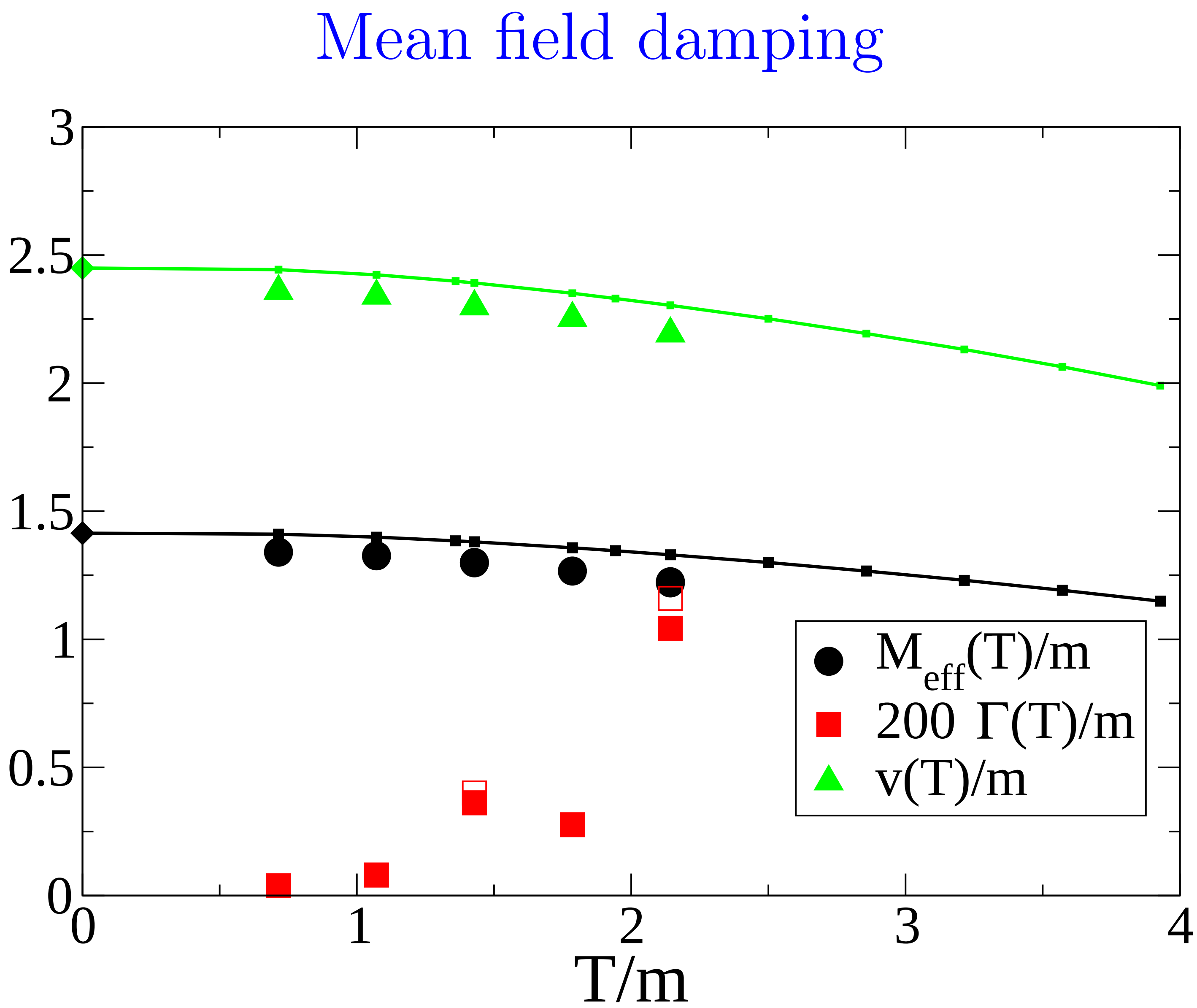


Equilibration



Mean field damping




Mean field damping
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4
T/m
$M_{eff}(T)/m$
200 $\Gamma(T)/m$
v(T)/m

Conclusion

- In the symmetric phase, $\mathrm{Re}\Sigma(\lambda^2)/\mathrm{Re}\Sigma(\lambda) \simeq 0.3$ at $\lambda = 6$. Expansion will break down for much larger λ. Renormalization.
- $\mathrm{Im}\Sigma(\lambda^2)$ is slightly larger than non-resummed analytical calculations.
- In the broken phase, resummation introduces $\mathrm{Im}\Sigma \neq 0$ at $O(\lambda)$. Qualitatively different from perturbation theory.
- Equilibration time: $O(\lambda^2)/O(\lambda) \sim O(1)$.
- $O(\lambda)$ does not dominate $O(\lambda^2)$. Probably because first level of resummation is $O(\lambda^4 v^4) = O(\lambda^2)$.